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A MODEL OF A NON-HOMOGENEOUS PSEUDOFLUIDIZED LAYER WITH PARTICLE EXCHANGE BETWEEN THE NON-HOMOGENEITY AND THE LAYER*

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A non-stationary model of a non-homogeneous pseudofluidized layer with local solid-phase non-homogeneities moving in the continuous layer /1/ is considered. Contrary to the previous studies /1, 2/, we assume that the solid-phase mass locked in the spherical packet of particles modelling the non-homogeneity may vary through the influx of particles into the packet or efflux of particles from the packet into the continuous phase of the layer. Particle concentrations inside and outside the packet remain constant. We assume that the solid-phase density is high compared with the density of the fluidizing agent (e.g., solid particles suspended in a gas stream), and the interaction between the phases is linear with respect to the relative velocity of the particles. In this formulation, the problem is similar to the growth (dissolution) of bubbles in a liquid and of drops in a liquid or a gas /3-5/.

The purpose of the study is to derive a system of equations linking the dynamics of the local solid-phase non-homogeneity with the velocity of its motion in the pseudofluidized layer. Packet "lifetime" is estimated. Some examples are considered.

1. Statement of the problem. We introduce a non-inertial spherical system of coordinates attached to the centre of a packet of radius $a(t)$ with the polar axis aligned in the direction of the vector of gravitational acceleration (Fig.1); the velocity of the packet in the laboratory system $x_1O_1y_1$ is $\mathbf{U}_d(t)$. In the light of the assumptions listed above, using a model of interpenetrating ideal fluids, we write the system of equations of motion and continuity of the fluid and solid phases inside and outside the packet:

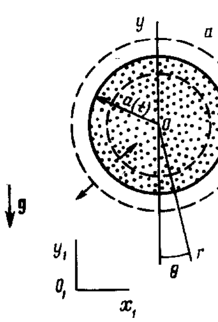

Fig.1

$$r > a(t),\ \mathbf{v}(\mathbf{r},t) - \mathbf{w}(\mathbf{r},t) = -k(\varepsilon)\nabla p_f(\mathbf{r},t) \qquad (1.1)$$
$$\nabla\mathbf{v}(\mathbf{r},t) = 0,\ d_s\rho\,[\partial/\partial t + \mathbf{w}(\mathbf{r},t)\nabla]\,\mathbf{w}(\mathbf{r},t) = -\nabla[p_f(\mathbf{r},t) + p_s(\mathbf{r},t)] \pm d_s\rho U_d^{\boldsymbol{\cdot}}(t)\,\mathbf{g}/g + d_s\rho\mathbf{g}$$
$$\nabla\mathbf{w}(\mathbf{r},t) = 0,\ \varepsilon + \rho = 1$$
$$r < a(t),\ \mathbf{v}'(\mathbf{r},t) - \mathbf{w}'(\mathbf{r},t) = -k'(\varepsilon')\nabla p_f'(\mathbf{r},t)$$
$$\nabla\mathbf{v}'(\mathbf{r},t) = 0,\ d_s\rho'\,[\partial/\partial t + \mathbf{w}'(\mathbf{r},t)\nabla]\,\mathbf{w}'(\mathbf{r},t) = -\nabla[p_f'(\mathbf{r},t) + p_s'(\mathbf{r},t)] \pm d_s\rho' U_d'(t)\,\mathbf{g}/g + d_s\rho'\mathbf{g}$$
$$\nabla\mathbf{w}'(\mathbf{r},t) = 0,\ \varepsilon' + \rho' = 1,\ U_d(t) = |\mathbf{U}_d(t)|$$

Here $\mathbf{v}, \mathbf{w}, p_f, p_s, \varepsilon, \rho$ are the locally averaged velocities, pressures, and volume concentrations of the fluidizing agent (of density d_f) and the dispersed solid particles (of density d_s, $d_f/d_s \ll 1$), respectively, $k(\varepsilon)$ is the perviousness of the pseudofluidized layer and $\mathbf{g}$ is the gravitational acceleration. The prime denotes the parameters of the two-phase flow inside the packet. The inertia forces acting on a unit solid-phase volume in the non-inertial coordinate system are proportional to $\pm U_d^{\boldsymbol{\cdot}}(t)$ and are allowed for in the third and seventh equations in (1.1). Here and in what follows, unless otherwise specified, the top sign corresponds to a rising packet $(\rho' < \rho)$ and the bottom sign to a sinking packet $(\rho' > \rho)$.

In the non-inertial coordinate system, the boundary conditions on the moving surface of the packet are written for the axisymmetric case in the form /1/

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$$r = a(t),\ \rho(w_r - D_r) = \rho'(w_r' - D_r),\ \varepsilon(v_r - D_r) = \varepsilon'(v_r' - D_r) \tag{1.2}$$
$$p_f = p_f',\ p_s' - p_s = d_s[\rho(w_r - D_r)^2 - \rho'(w_r' - D_r)^2]$$

The first two equations are expressions of flux conservation of solid and fluid phases at the discontinuity; the last two equations are the balance conditions of the normal stresses in the fluid and solid phases, respectively. In (1.2), D is the velocity of motion of the discontinuity (the packet surface) in a coordinate system attached to the packet centre. The equation of the discontinuity surface in this system has the form $F(r, \theta, \varphi, t) = r - a(t) = 0$. Therefore, $\mathbf{D} = -\mathbf{i}_r(\partial F/\partial t)/|\nabla F| = \mathbf{i}_r a^{\cdot}(t)$, where $\mathbf{i}_r$ is the unit vector in the radial direction. Hence $D_r = a^{\cdot}(t)$.

In what follows, we assume that the dispersed-phase particles in the internal flow region $r < a(t)$ (i.e., the particles which at the current instant of time are enclosed by the packet boundary) are at rest relative to the boundary, i.e., $w_r'|_{r=a(t)} = 0$. At the same time, the velocity of the dispersed phase normal to the boundary $w_r|_{r=a(t)}$ is non-zero outside the packet and it determines the rate of "growth" and "dissolution" of the packet in the ambient layer. This assumption implies that there are no solid-phase sources (sinks) inside the packet, whose density is constant.

Under these assumptions, the surface density of the dispersed-phase concentration discontinuity is thus constant, and the discontinuity mass increases for a growing packet and decreases for a dissolving packet. In the model previously conisdered /1/, the surface density of the discontinuity varied due to variation of the packet size, with the packet mass remaining constant $(w_r|_{r=a(t)} = w_r'|_{r=a(t)})$.

Using our assumption and the expression for the velocity of the discontinuity D, we rewrite conditions (1.2) in the form

$$r = a(t),\ \rho(a^{\cdot} - w_r) = \rho' a^{\cdot},\ \varepsilon(v_r - a^{\cdot}) = \varepsilon'(v_r' - a^{\cdot}) \tag{1.3}$$
$$p_f = p_f',\ p_s' - p_s = d_s[\rho(w_r - a^{\cdot})^2 - \rho' a^{\cdot 2}] =$$
$$-d_s\rho' w_r a^{\cdot} = d_s \rho' a^{\cdot 2}(\rho'/\rho - 1)$$

The boundary conditions (1.3) on the packet surface should be supplemented with flow homogeneity conditions of the fluid and solid phases far from the packet and the conditions of bounded flow velocities throughout the entire region.

2. Velocity fields and pressure distribution of the phases inside and outside the packet. Consider the case when the flow of the solid phase outside the packet is potential, i.e., $\mathbf{w}(\mathbf{r}, t) = \nabla\varphi_s(\mathbf{r}, t)$, where $\varphi_s(\mathbf{r}, t)$ is the velocity potential. From the fourth equation in (1.1) it follows that the potential is a harmonic function at each instant of time

$$\Delta\varphi_s(\mathbf{r}, t) = 0 \tag{2.1}$$

and it satisfies the conditions

$$\partial\varphi_s(\mathbf{r}, t)/\partial r|_{r=a(t)} = w_r(\mathbf{r}, t)|_{r=a(t)} = a^{\cdot}(t)(1 - \rho'/\rho) \tag{2.2}$$
$$r \to \infty,\ \varphi_s(\mathbf{r}, t) \to \varphi_s^{\circ}(\mathbf{r}, t)|$$

where $\varphi_s^{\circ}(\mathbf{r}, t)$ is the potential of the ideal fluid flow past a sphere of variable radius $a(t)$ which is homogeneous at infinity.

The solution of problem (2.1), (2.2) has the form

$$\varphi_s(\mathbf{r}, t) = \pm U_d(t)\left[1 + \frac{a^3(t)}{2r^3}\right] r\cos\theta - \frac{a^2(t)\, a^{\cdot}(t)(\rho - \rho')}{r\rho} \tag{2.3}$$

From the continuity equation of the solid phase inside the packet we conclude that the vector $\mathbf{w}'(\mathbf{r}, t)$ is solenoidal and the stream function $\psi_s'(\mathbf{r}, t)$ can be introduced for the internal flow in a standard way:

$$w_r'(\mathbf{r}, t) = \frac{1}{r^2\sin\theta}\frac{\partial\psi_s'(\mathbf{r}, t)}{\partial\theta},\ w_\theta'(\mathbf{r}, t) = -\frac{1}{r\sin\theta}\frac{\partial\psi_s'(\mathbf{r}, t)}{\partial r}$$

Taking the curl of both sides of the equation of motion of the solid phase inside the packet, we obtain the following equation for the stream function describing internal flow (D^2 is the Stokes operator):

$$\frac{\partial}{\partial t}(D^2\psi_s') + \frac{1}{r^2}\frac{\partial(\psi_s', D^2\psi_s')}{\partial(r,\mu)} - \frac{2}{r^2}D^2\psi_s' L\psi_s' = 0 \tag{2.4}$$

$$D^2 = \frac{\partial^2}{\partial r^2} + \frac{1-\mu^2}{r^2}\frac{\partial^2}{\partial\mu^2}, \quad L = \frac{\mu}{1-\mu^2}\frac{\partial}{\partial r} + \frac{1}{r}\frac{\partial}{\partial\mu}, \quad \mu = \cos\theta$$

Below we will consider the simplest partial solution of Eq.(2.4) $\psi_s'(\mathbf{r}, t) = \text{const}$, which corresponds to the case when the particles are at rest inside the packet:

$$r < a(t), \ \mathbf{w}'(\mathbf{r}, t) = 0 \tag{2.5}$$

The solid phase slips through the porous body boundary $r \leqslant a(t)$, so that $w_\theta|_{r=a(t)} \neq 0$.

Let us now determine the pressure fields of the solid and fluid phases. Applying the divergence to the equations of fluid-phase motion and using the equations of continuity, we obtain for the corresponding pressures inside and outside the packet

$$r > a(t), \ \Delta p_f(\mathbf{r}, t) = 0; \ r < a(t), \ \Delta p_f'(\mathbf{r}, t) = 0 \tag{2.6}$$

The solutions of Eqs.(2.6) must satisfy the second and fourth conditions (1.3) on the packet surface. Moreover, by the homogeneity of the layer far from the packet, the pressure $p_f(\mathbf{r}, t)$ should satisfy the condition $\partial p_f(\mathbf{r}, t)/\partial y|_{r\to\infty} = -v_0/k(\varepsilon)$ (v_0 is the pseudofluidization rate and $y = -r\cos\theta$ is the vertical coordinate), and due to bounded velocities of the phases inside the packet $|p_f'(\mathbf{r}, t)| < \infty$.

The solution of the first Eq.(2.6) can be represented as a series in Legendre polynomials

$$p_f(\mathbf{r}, t) = \frac{v_0}{k(\varepsilon)} r\cos\theta + \sum_{n=1}^{\infty} B_n(t) P_{n-1}(\cos\theta) r^{-n} a^{n+1}(t) + p_{f\infty}(t) \tag{2.7}$$

Here $p_{f\infty}(t)$ is the fluid-phase pressure in the equatorial plane of the packet at a large distance.

A similar expansion for the function $p_f'(\mathbf{r}, t)$ without singularities for $r < a(t)$ can be represented in the form $|p_f'(\mathbf{r}, t)| < \infty$.

$$p_f'(\mathbf{r}, t) = \sum_{n=0}^{\infty} A_n(t) P_n(\cos\theta) a^{1-n}(t) r^n \tag{2.8}$$

The unknown coefficients $B_n(t)$, $A_n(t)$ in relationships (2.7) and (2.8) should be determined from the boundary conditions.

The pressure distribution of the fluidizing agent in the entire flow region has the form

$$r > a(t), \quad p_f(\mathbf{r}, t) = \frac{v_0}{k(\varepsilon)} r\cos\theta + B_1(t)\frac{a^2(t)}{r} + B_2\frac{a^3(t)}{r^2}\cos\theta + p_{f\infty}(t) \tag{2.9}$$

$$r < a(t), \quad p_f'(\mathbf{r}, t) = B_1(t)a(t) + \frac{3v_0}{k}\frac{\varepsilon k}{2\varepsilon k + \varepsilon' k'} r\cos\theta + p_{f\infty}(t)$$

$$B_1(t) = \frac{\varepsilon\rho' - \varepsilon'\rho}{\rho}\frac{1}{\varepsilon k}a^{\cdot}(t), \quad B_2 = \frac{v_0}{k}\frac{\varepsilon k - \varepsilon' k'}{2\varepsilon k + \varepsilon' k'}$$

In the stationary approximation, $B_1(t) = 0$ and our expressions are identical with those of /2/.

The velocity fields of the solid phase are determined from relationships (2.9), (2.3) and the corresponding equations of motion in the form $\mathbf{v}(\mathbf{r}, t) = \mathbf{w}(\mathbf{r}, t) - k(\varepsilon)\nabla p_f(\mathbf{r}, t)$, or in coordinate form

$$v_r(r, t) = \pm U_d(t)\left[1 - \frac{a^3(t)}{r^3}\right]\cos\theta + \frac{a^2(t)a^{\cdot}(t)(\rho - \rho')}{r^2\rho} - \left[v_0 - 2kB_2\frac{a^3(t)}{r^3}\right]\cos\theta + kB_1(t)\frac{a^2(t)}{r^2} \tag{2.10}$$

$$v_\theta(\mathbf{r}, t) = \mp U_d(t)\left[1 + \frac{a^3(t)}{2r^3}\right]\sin\theta + \left[v_0 + kB_2\frac{a^3(t)}{r^3}\right]\sin\theta$$

Similarly, $\mathbf{v}'(\mathbf{r}, t) = -k'(\varepsilon')\nabla p_f'(\mathbf{r}, t)$ or

$$v_r'(\theta) = -3v_0\frac{\varepsilon k'}{2\varepsilon k + \varepsilon' k'}\cos\theta, \quad v_\theta'(\theta) = 3v_0\frac{\varepsilon k'}{2\varepsilon k + \varepsilon' k'}\sin\theta \tag{2.11}$$

Expressions (2.11) describe the homogeneous steady flow of the fluid phase $|v'| = 3v_0\varepsilon k' (2\varepsilon k + \varepsilon' k')^{-1}$ filtering through the packet interior from the bottom upwards, irrespective of whether the packet is rising $(\rho' < \rho)$ or sinking $(\rho' > \rho)$.

Let us now consider the pressure fields of the dispersed phase inside and outside the packet. In the external flow region, the equation of motion of the solid phase has a Cauchy-Lagrange integral in the form

$$d_s\rho\partial\varphi_s(\mathbf{r}, t)/\partial t + {}^1/_2 d_s\rho\mathbf{w}^2(\mathbf{r}, t) + p_f(\mathbf{r}, t) + p_s(\mathbf{r}, t) - d_s\rho[\pm U_d^{\cdot}(t)/g + 1]\,\mathbf{gr} = {}^1/_2 d_s\rho U_d^2(t) + p_{f\infty}(t) + p_{s\infty} \tag{2.12}$$

From (2.12) we obtain the pressure distribution of the solid phase outside the packet $(r \geqslant a(t))$

$$p_s(\mathbf{r}, t) = {}^1/_2 d_s\rho[U_d^2(t) - \mathbf{w}^2(\mathbf{r}, t)] - d_s\rho\partial\varphi_s(\mathbf{r}, t)/dt + p_{f\infty}(t) + p_{s\infty} - p_f(\mathbf{r}, t) + d_s\rho[\pm U_d^{\cdot}(t)/g + 1]\,\mathbf{gr} \tag{2.13}$$

In (2.12) and (2.13), $p_{s\infty}$ is the pressure of the dispersed phase in the homogeneous layer region far from the packet; the velocity potential $\varphi_s(\mathbf{r}, t)$ is determined by (2.3).

Thus, the pressure distribution $p_s(\mathbf{r}, t)$ on the surface of the packet is given by

$$\begin{gathered} p_s(\mathbf{r}, t)|_{r=a(t)} = {}^1/_2 d_s\rho[U_d^2(t) - a^{\cdot 2}(t)(1 - \rho'/\rho)^2 - {}^9/_4\sin^2\theta U_d^2(t)] - \\ d_s\rho[\pm{}^3/_2 U_d^{\cdot}(t)\,a(t)\cos\theta \pm {}^3/_2 U_d(t)\,a^{\cdot}(t)\cos\theta - 2(1 - \rho'/\rho)\,a^{\cdot 2}(t) - (1 - \rho'/\rho)\,a(t)\,a^{\cdot\cdot}(t)] + \\ p_{f\infty}(t) + p_{s\infty} - p_f(\mathbf{r}, t)|_{r=a(t)} + d_s\rho[\pm U_d^{\cdot}(t)/g + 1]\,ga(t)\cos\theta \end{gathered} \tag{2.14}$$

The equation of motion of the disperse phase inside the packet for $\mathbf{w}' = 0$ also has a Cauchy-Lagrange integral in the form

$$p_s'(\mathbf{r}, t) = -p_f'(\mathbf{r}, t) + d_s\rho'[\pm U_d^{\cdot}(t)/g + 1]\,\mathbf{gr} + p_{s0}'(t) \tag{2.15}$$

whence we obtain for the pressure distribution on the surface of the packet

$$p_s'(\mathbf{r}, t)|_{r=a(t)} = -p_f'(\mathbf{r}, t)|_{r=a(t)} + d_s\rho'[\pm U_d^{\cdot}(t)/g + 1]\,ga(t)\cos\theta + p_{s0}'(t) \tag{2.16}$$

The function $p_{s0}'(t)$ in (2.15) and (2.16) is defined by the condition $p_s'(\mathbf{r}, t)|_{\rho'=0} = 0$, which, combined with the second equality in (2.9), gives the relationship

$$p_{s0}'(t) = -(\varepsilon k)^{-1} a(t)\,a^{\cdot}(t) + p_{f\infty}(t)$$

3. *The velocity of the packet and the evolution of the packet size.* The need to satisfy condition (1.3) for the dispersed-phase pressure jump globally along the entire packet boundary contradicts the assumption of a spherical packet shape. Following the Davies-Taylor procedure /6/, we restrict the analysis to the sections of the packet surface that are immediately adjacent to the frontal critical point ($r = a(t)$, $\theta = \pi$ for a rising packet and $r = a(t)$, $\theta = 0$ for a sinking packet), where condition (1.3) can be satisfied locally. To this end we take $\sin^2\theta|_{\theta\to 0,\pi} = \delta$ and therefore $\cos\theta = \mp 1 \pm \delta/2 + O(\delta^2)$. Retaining terms of the zeroth and first order in the expansion of the dispersed-phase pressure discontinuity on the packet boundary in powers of δ and using relationships (2.14), (2.15), we obtain a system of ordinary differntial equations

$$\begin{gathered} {}^1/_2(\lambda - 1)(3 - \lambda)\,a^{\cdot 2}(t) + (\lambda - 1)\,a(t)\,a^{\cdot\cdot}(t) - p_{s\infty}/(d_s\rho) + {}^7/_4 U_d^2(t) = a(t)\,a^{\cdot}(t)/(\varepsilon k\rho d_s) \\ {}^3/_2 a^{\cdot}(t)\,U_d(t) + {}^1/_2 a(t)\,U_d^{\cdot}(t)(1 + 2\lambda) + {}^9/_4 U_d^2(t) = \mp(\lambda - 1)\,ga(t), \quad \lambda = \rho'/\rho \end{gathered} \tag{3.1}$$

connecting the velocity of the packet in the layer $U_d(t)$ with the rate of change of the packet size $a^{\cdot}(t)$. In the limiting case $\lambda = 0$ (no particles inside the packet), Eqs.(3.1) are identical in the linear phase interaction approximation with the corresponding equations of motion and growth previously obtained for a bubble /7/.

Eqs.(3.1) have a steady-state solution

$$U_{d*} = 2\left(\frac{p_{s\infty}}{7d_s\rho}\right)^{1/2}, \quad a_* = \mp\frac{1}{\lambda-1}\,\frac{9}{7}\,\frac{p_{s\infty}}{d_s\rho g} \tag{3.2}$$

which in the limit as $\lambda \to 0$ corresponds to the results obtained in /7, 8/.

From (3.2) it follows that the steady-state velocity of the packet is independent of the particle concentration inside the packet, whereas the steady-state packet size essentially depends on the relative packet density λ and may vary over a wide range:

$$\min\left\{1, \frac{\rho}{1-\rho}\right\} \leqslant \bar{a}_* < \infty, \quad \bar{a}_* = a_*\left(\frac{9}{7}\,\frac{p_{s\infty}}{d_s\rho g}\right)^{-1}$$

For any $\bar{a}_* > \max\{1, \rho/(1-\rho)\}$, both thin and dense packets with relative densities given respectively by

$$\lambda_{1*} = (\bar{a}_* - 1)/\bar{a}_* \text{ and } \lambda_{2*} = (\bar{a}_* + 1)/\bar{a}_*$$

have the same steady-state radius a_*.

For a bubble, the steady-state is known to be unstable /7, 8/, i.e., for $a(t) > a_*$ the bubble grows $(a^{\cdot}(t) > 0)$ and for $a(t) < a_*$ it collapses $(a^{\cdot}(t) < 0)$. We naturally expect to obtain the same results for a packet of particles.

Eliminating $U_d(t)$ from Eqs.(3.1), we obtain the following equation describing the evolution of a three-dimensional packet in our model:

$$\begin{gathered} a^{\cdots} = \left(\lambda - 4 - \frac{6}{1+2\lambda}\right)\frac{a^{\cdot}a^{\cdot\cdot}}{a} - \frac{(\lambda + {}^7/_2)\,\eta}{(1-\lambda)(\lambda + {}^1/_2)}\,\frac{a^{\cdot 2}}{a} - \frac{\eta}{1-\lambda}\,a^{\cdot\cdot} + \\ \frac{21}{2(1+2\lambda)(\lambda-1)}\,\frac{a^{\cdot}}{a^2} - \frac{3(3-\lambda)}{1+2\lambda}\,\frac{a^{\cdot 3}}{a^2} + \\ \frac{63}{4(1+2\lambda)(1-\lambda)a^2}\left[1 - \frac{4}{7}(\lambda-1)\,aa^{\cdot\cdot} - \frac{2}{7}(\lambda-1)(3-\lambda)\,a^{\cdot 2} + \right. \\ \left.\frac{4}{7}\,\eta aa^{\cdot}\right]^{1/2}\left[\mp(\lambda-1)\,a - 1 + \frac{4}{7}(\lambda-1)\,aa^{\cdot\cdot} + \right. \\ \left.\frac{2}{7}(\lambda-1)(3-\lambda)\,a^{\cdot 2} - \frac{4}{7}\,\eta aa^{\cdot}\right], \quad \lambda \neq 1 \\ \eta = \frac{1}{18}\,\frac{C_1\rho}{\varepsilon^3}\,\frac{t_0}{\tau_0}, \quad \tau_0 = \frac{2}{9}\,\frac{a_p^2 d_s}{d_f\nu_f} \end{gathered} \tag{3.3}$$

Here we use dimensionless variables relative to the following scales: length $a_{*b} = {}^9/_7\, p_{s\infty}/(d_s\rho g)$ (the steady-state bubble radius) and time $t_0 = {}^3/_2\,(a_{*b}/g)^{1/2}$. The same symbols are used to denote the dimensionless packet radius and dimensionless time. The dimensionless parameter η characterizes the ratio of time macroscales and microscales of non-stationarity, $C_1 \approx 150$ is the Ergan constant and τ_0 is the relaxation time of the velocity of a solid particle with density d_s and radius a_p in a viscous gas of density d_f and kinematic viscosity ν_f in the Stokes approximation.

The ordinary differential Eq.(3.3) corresponds to a third-order dynamic autonomous system, which may be investigated by the general methods of the theory of non-linear oscillations /9, 10/. From (3.3) it follows that the motion of the representative point in phase space $(a, a^{\cdot}, a^{\cdot\cdot})$ of the system is restricted to the region where

$$1 - {}^4/_7\,(\lambda-1)\,aa^{\cdot\cdot} - {}^2/_7\,(\lambda-1)(3-\lambda)\,a^{\cdot 2} + {}^4/_7\,\eta aa^{\cdot} \geqslant 0$$

The motion of the phase point on the boundary of this region corresponds to unsteady evolution of an immobile $(U_d = 0)$ packet of particles in the layer. From the second equation of (3.1) it follows that this case is realizable only for $\lambda = 1$, i.e., when the volume concentrations of the solid phase inside and outside the packet are equal, which corresponds to the absence of inhomogeneity in the layer.

Let us linearize Eq.(3.3) near the steady-state solution, setting $a(t) = a_* + \Delta(t)$, where $\Delta(t)$ is a small deviation of the dimensionless packet radius from the equilibrium value $a_* = \mp 1/(\lambda - 1)$.

The linear approximation equation has the form

$$\Delta^{\cdots} + \left[\frac{\eta}{1-\lambda} \mp \frac{9(\lambda-1)}{1+2\lambda}\right]\Delta^{\cdot\cdot} + \left[\frac{21}{2}\,\frac{1-\lambda}{1+2\lambda} \pm \frac{9\eta}{1+2\lambda}\right]\Delta^{\cdot} \mp \frac{63}{4}\,\frac{(\lambda-1)^2}{1+2\lambda}\,\Delta = 0 \tag{3.4}$$

In the limiting case when $\lambda = 0$ $(\rho' = 0)$, relationship (3.4) is identical with the previous equation /7/ that describes the evolution of a bubble near equilibrium in the linear phase interaction approximation.

The characteristic polynomial of Eq.(3.4) is unstable for all λ. In particular, for $\lambda < 1$ it has precisely one positive root. This means that, as for a bubble, the steady-state of a particle packet with any relative density λ is unstable: for $a > a_*$ there is a regular inflow of the dispersed phase from the layer into the packet and its mass increases; for $a < a_*$ the particles escape into the layer and the packet "dissolves".

4. Evolution of a two-dimensional circular packet of particles in a pseudofluidized layer. The problem of the motion of a spherical packet of particles in a pseudofluidized layer may be extended to the case of a cylindrical packet. As previously, we consider the simplest case when there is no relative motion of dispersed-phase particles comprising the inhomogeneity (condition (2.5)). The surface densities of particles outside and inside the packet are equal to the corresponding volume densities ρ and ρ' /11/.

An essential difference between the plane and the three-dimensional case is the non-unique solvability of the corresponding boundary-value problems. This necessitates supplementing the equations of packet motion, similar to Eqs.(3.1), with additional conditions in order to ensure unique solvability.

As in the three-dimensional case, the flow field of the dispersed phase outside the packet is described by Eq.(2.1) with boundary conditions (2.2). The solution of problem (2.1), (2.2) has the form

$$\varphi_s(\mathbf{r}, t) = \pm U_d(t)\,[r + a^2(t)/r]\cos\theta + a(t)\,a^{\cdot}(t)\,(1-\lambda)\ln(r/L(t)) \qquad (4.1)$$

Here $L(t)$ is an arbitrary function of time having the dimensions of length, which does not vanish anywhere in its domain of definition. The velocity field $\mathbf{w}(\mathbf{r}, t)$ is obviously independent of $L(t)$.

The pressure distribution of the fluid phase outside and inside the packet is described in our model by the harmonic functions $p_f(\mathbf{r}, t)$ and $p_f'(\mathbf{r}, t)$ (see (2.6)), which in the plane problem are sought in series form (r and θ are the cylindrical coordinates)

$$p_f(\mathbf{r}, t) = \frac{v_0}{k(\varepsilon)}\, r\cos\theta + \sum_{n=1}^{\infty} \frac{a^{n+1}(t)}{r^n}\,[A_n(t)\cos n\theta + B_n(t)\sin n\theta] + C(t)\ln\frac{r}{L(t)} + p_{f\infty}(t) \qquad (4.2)$$

$$p_f'(\mathbf{r}, t) = \sum_{n=0}^{\infty} a^{1-n}(t)\, r^n\,[A_n'(t)\cos n\theta + B_n'(t)\sin n\theta]$$

Imposing the boundary conditions (1.3) on (4.2), we determine the coefficients $A_n(t)$, $B_n(t)$, $A_n'(t)$, $B_n'(t)$, $C(t)$ and finally obtain

$$r > a(t),\quad p_f(\mathbf{r}, t) = \frac{v_0}{k}\left[r + \frac{\varepsilon k - \varepsilon' k'}{\varepsilon k + \varepsilon' k'}\,\frac{a^2(t)}{r}\right]\cos\theta - \frac{\varepsilon\rho' - \varepsilon'\rho}{\varepsilon\rho k}\, a(t)\,a^{\cdot}(t)\ln\frac{r}{L(t)} + p_{f\infty}(t) \qquad (4.3)$$

$$r < a(t),\quad p_f'(\mathbf{r}, t) = \frac{2v_0\varepsilon}{\varepsilon k + \varepsilon' k'}\, r\cos\theta - \frac{\varepsilon\rho' - \varepsilon'\rho}{\varepsilon\rho k}\, a(t)\,a^{\cdot}(t)\ln\frac{a(t)}{L(t)} + p_{f\infty}(t)$$

The non-unique solvability of problem (2.6), (1.3) does not affect the velocity fields of the phases, because according to the original Eqs.(1.1) these fields are determined by the space derivatives of expressions (4.3).

The pressure distribution of the fluid phase obtained in this way also satisfies the condition of constancy of the gradient of the function $p_f(\mathbf{r}, t)$ far from the packet in the homogeneous layer and has no singularities in the internal flow region $r < a(t)$.

The equations of motion of the solid phase outside and inside the packet, as in the three-dimensional case, have a Cauchy-Lagrange integral. Using expressions (4.1) and (4.2), we find the pressure field of the dispersed phase in the entire flow region in the form

$$r > a(t),\quad p_s(\mathbf{r}, t) = {}^1/_2 d_s\rho\,[U_d^2(t) - \mathbf{w}^2(\mathbf{r}, t)] - d_s\rho\,\partial\varphi_s(\mathbf{r}, t)/\partial t + p_{f\infty}(t) + p_{s\infty} - p_f(\mathbf{r}, t) + d_s\rho\,[\pm U_d^{\cdot}(t)/g + 1]\,\mathbf{gr} \qquad (4.4)$$

$$r < a(t),\quad p_s'(\mathbf{r}, t) = -p_f'(\mathbf{r}, t) + d_s\rho'\,[\pm U_d^{\cdot}(t)/g + 1]\,\mathbf{gr} + p_{f\infty}(t) + (\varepsilon k)^{-1} a(t)\,a^{\cdot}(t)\ln[a(t)/L(t)]$$

The second relationship in (4.4) satisfies the condition $p_s'(\mathbf{r}, t)|_{\rho'\to 0} \to 0$, which has been previously used in (2.16).

From the results of (4.3) and (4.4) it follows that the effect of an infinitely long cylindrical non-homogeneity on distant parts of the layer in the plane problem is more pronounced than in the three-dimensional case: the pressure perturbation in the fluid phase in the homogeneous layer logarithmically increases at infinity in the equatorial plane of the packet. Similarly, from the first equality in (4.4) we obtain that the pressure perturbation of the dispersed phase also logarithmically increases everywhere in the layer with distance from the packet. For a spherical packet occupying a bounded region, such pressure perturbations, on the contrary, are vanishingly small far from the packet. These pressure perturbations do not affect the flow fields of the phases at infinity because the velocities $\mathbf{v}, \mathbf{w}$ are determined by the pressure gradients of the phases, whose values far from the packet, as in the three-dimensional case, correspond to the homogeneous layer:

$$\nabla p_f(\mathbf{r}, t)|_{r\to\infty} = \frac{v_0}{k(\varepsilon)}\frac{\mathbf{g}}{g}, \quad \nabla p_s(\mathbf{r}, t)|_{r\to\infty} = 0$$

Thus, $p_{f\infty}(t)$ and $p_{s\infty}$ in the plane problem correspond to the pressure of the fluid phase in the equatorial plane $\theta = \pm\pi/2$ for its motion with velocity $\mathbf{U_d}(t)$ in the laboratory system and the pressure of the solid phase when there is no inhomogeneity in the layer.

Now, relying on relationships (4.4) and expanding the corresponding expressions for the pressure discontinuity of the solid phase on the packet surface in the neighbourhood of frontal points in series by the Davies-Taylor method, we obtain a system of differential equations describing the evolution of a plane circular packet:

$$\begin{gathered}\frac{7}{2}U_d^2(t) + a^{\cdot 2}(t)(1-\lambda)\left[\frac{1}{2} + \frac{\lambda}{2} - \frac{a(t)L^{\cdot}(t)}{a^{\cdot}(t)L(t)}\right] + \\ \left\{(1-\lambda)[a^{\cdot 2}(t) + a(t)a^{\cdot\cdot}(t)] + \frac{a(t)a^{\cdot}(t)}{\varepsilon k\rho d_s}\right\}\ln\frac{a(t)}{L(t)} - \frac{p_{s\infty}}{d_s\rho} = 0 \\ (\lambda+1)\,U_d^{\cdot}(t)\,a(t) + 2U_d(t)\,a^{\cdot}(t) + 4U_d^2(t) = \mp(\lambda-1)\,ga(t) \\ \lambda = \rho'/\rho\end{gathered} \quad (4.5)$$

The relationships $L(t)$ is specified assuming that a packet whose density is equal to that of the surrounding layer remains immobile, i.e., $U_d(t) = 0$ for $\lambda = 1$. In this case, the second equation of system (4.5) is identically satisfied and from the first equation we have

$$L(t) = a(t)\exp\left(-\frac{\varepsilon k p_{s\infty}}{a(t)a^{\cdot}(t)}\right) \quad (4.6)$$

Using the equality (4.6), the equations describing the evolution of the packet are written in the form

$$\begin{gathered}7U_d^2(t) - (\lambda-1)^2 a^{\cdot 2}(t) = 0 \\ (\lambda+1)\,U_d^{\cdot}(t)\,a(t) + 2U_d(t)\,a^{\cdot}(t) + 4U_d^2(t) = \\ \mp(\lambda-1)\,ga(t)\end{gathered} \quad (4.7)$$

From relationships (4.7) it follows that for packets with slowly varying velocity (so that $U_d^{\cdot}(t) \ll g$) we always have $U_d^2(t) \sim ga(t)$, which, in particular, is consistent with numerous experimental measurements of the velocity of bubbles in pseudofluidized systems /12, 13/.

In the limiting case when $\rho = 0$ (a bunch with particle density ρ' in a pure gas) we obtain from Eqs.(4.7) $a^{\cdot} = 0$, $U_d^{\cdot} = g$. The particle bunch preserves its size in this case and moves through the gas just like a falling body.

System (4.7) can be investigated by the methods of the theory of non-linear oscillations.

From the first equation of the system we obtain

$$U_d(t) = \frac{1}{\sqrt{7}}|\lambda-1|\,|a^{\cdot}(t)|, \quad U_d^{\cdot}(t) = \frac{1}{\sqrt{7}}|\lambda-1|\,H(a^{\cdot})\,a^{\cdot\cdot}(t)$$

where $H(a^{\cdot})$ is the Heaviside function.

Substituting the expression for $U_d(t)$ and $U_d^{\cdot}(t)$ into the second equation of the system, we obtain the following autonomous second-order ordinary differential equation for the packet radius:

$$\frac{\lambda+1}{\sqrt{7}} H(a^{\cdot})\, a^{\cdot\cdot}(t)\, a(t) \pm \frac{2}{\sqrt{7}} |a^{\cdot}(t)|\, a^{\cdot}(t) \mp \frac{4}{7} (\lambda-1)\, a^{\cdot 2}(t) = g a(t) \tag{4.8}$$

The upper sign, as usual, corresponds to rising packets $(\lambda < 1)$ and the lower sign to sinking packets $(\lambda > 1)$.

The phase spaces $(a, a^{\cdot})$ of Eq.(4.8) are bivalent: the evolution equations are different for growing packets $(a^{\cdot}(t) > 0,\ H(a^{\cdot}) = 1)$ and dissolving packets $(a^{\cdot}(t) < 0,\ H(a^{\cdot}) = -1)$ for any ratio of packet-to-layer densities. The halfline $a^{\cdot} = 0$ in the phase space separates regions with different behaviour of the phase trajectories.

As a result of this property, relation (4.8) can be represented as a system of equations

$$\lambda < 1,\ a^{\cdot\cdot}(t)a(t) + \beta_+(\lambda)a^{\cdot 2}(t) = \alpha(\lambda)a(t),\ a^{\cdot}(t) > 0 \tag{4.9}$$
$$a^{\cdot\cdot}(t)a(t) + \beta_-(\lambda)a^{\cdot 2}(t) = -\alpha(\lambda)a(t),\ a^{\cdot}(t) < 0$$
$$\lambda > 1,\ a^{\cdot\cdot}(t)a(t) + \beta_-(\lambda)a^{\cdot 2}(t) = \alpha(\lambda)a(t),\ a^{\cdot}(t) > 0 \tag{4.10}$$
$$a^{\cdot\cdot}(t)a(t) + \beta_+(\lambda)a^{\cdot 2}(t) = -\alpha(\lambda)a(t),\ a^{\cdot}(t) < 0$$
$$\left(\beta_\pm(\lambda) = \frac{2[\sqrt{7} \pm 2(1-\lambda)]}{\sqrt{7}(1+\lambda)},\quad \alpha(\lambda) = \frac{g\sqrt{7}}{1+\lambda}\right)$$

5. ***Integration of the equations describing packet evolution. Phase portraits.*** Consider the results obtained by integrating systems (4.9) and (4.10) separately for rising and sinking packets.

Rising packet $\lambda < 1$. The phase trajectories of Eqs.(4.9) are described by one-parameter families of curves

$$a^{\cdot}(a) = \left(C_{11}a^{-2\beta_+} + \frac{2\alpha}{2\beta_+ + 1}a\right)^{1/2},\quad a^{\cdot} > 0 \tag{5.1}$$
$$a^{\cdot}(a) = -\left(C_{12}a^{-2\beta_-} - \frac{2\alpha}{2\beta_- + 1}a\right)^{1/2},\quad a^{\cdot} < 0$$

where C_{11} and C_{12} are the parameters of the respective families, and $C_{12} \geqslant 0$.

The curves of the first family in (5.1) do not cross the axis for $C_{11} > 0$ and cross it once for $C_{11} < 0$. The parabola

$$a^{\cdot 2}(a) = \frac{2\alpha}{2\beta_+ + 1} a\ (C_{11} = 0) \tag{5.2}$$

separates the subsets of phase curves of these two types. All trajectories of the second family in (5.1) start on the axis $a^{\cdot} = 0$ and monotonically approach the axis $a = 0$ with time.

The phase space of system (4.9) is shown in Fig.2. The phase trajectories in both regions of continuous motion $a^{\cdot} > 0$ and $a^{\cdot} < 0$ are normal to the axis $a^{\cdot} = 0$ on the axis. The axis $a^{\cdot} = 0$ is therefore a continuum of unstable states of rest of the phase point $a = a_0$, $a_0^{\cdot} = 0$ (a_0, and $a_0^{\cdot}$ are the initial values of the radius and the rate of change of the radius of a circular packet). The phase point $(a, a^{\cdot})$ moves to one of the continuity regions as a result of any small deviation from the discontinuity line $a^{\cdot} = 0$ in the phase space. In general /10, 14/, the phase point may continuously pass through such discontinuity curves or surfaces and also move along them in a stable or unstable ("sliding") manner.

The phase point initially located in one of the regions $a^{\cdot} > 0$ or $a^{\cdot} < 0$ thus remains in that region, i.e., the mass of a growing packet will continue to increase, while a dissolving packet will continue to lose mass due to the efflux of particles to the exterior layer.

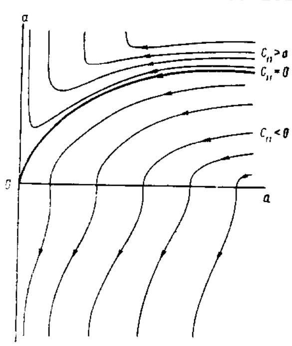

Fig.2

The growth region $a^{\cdot} > 0$ is partitioned by the curve (5.2) into two subregions: a packet with initial state above this curve in the phase plane grows non-monotonically. Specifically, the rate of growth at first decreases (very rapidly for small a_0) to some minimum value and then starts increasing. If the initial state $(a_0, a_0^{\cdot})$ lies on or below the parabola (5.2), the influx of the dispersed phase from the layer into the packet is monotonic.

For a dissolving packet, a useful characteristic of mass transfer of the solid phase from the packet to the layer is the packet "lifetime", which is measured from the initial state $a = a_0$, $a_0^{\cdot} = 0$:

$$T_*(\lambda) = \int_0^{T_*} dt = \int_{a_*}^{0} \frac{da}{a^{\cdot}(a)}$$

Using the second relation in (5.1), we obtain

$$T_*(\lambda) = \left(\frac{2\beta_- + 1}{2\alpha} a_0\right)^{1/2} I(\lambda), \quad I(\lambda) = \int_0^1 \frac{d\tau}{(\tau^{-2\beta_-} - \tau)^{1/2}}, \quad \tau = \frac{a}{a_0} \tag{5.3}$$

The corresponding value for a bubble is obviously $T_*(0)$. The improper integral $I(\lambda)$ with a singularity at $\tau = 1$ converges by the Cauchy criterion, because the integrand for $\tau \to 1$ is of order $\xi^{-1/2}$, where $\xi = 1 - \tau$.

Sinking packet $\lambda > 1$. Unlike the case of rising packets, the coefficient $\beta_+(\lambda)$ is sign-alternating for sinking packets when $a^\cdot < 0$. Specifically, $\beta_+(\lambda) \leqslant 0$ for $\lambda \geqslant \lambda_*$ and $\beta_+(\lambda) > 0$ for $\lambda < \lambda_*$, where $\lambda_* = 1 + \sqrt{7}/2$. This alters the appearance of the phase trajectories depending on the relative density of the dissolving packet $(a^\cdot < 0)$. It is also necessary to allow for the singularity in the second relation in (5.1) associated with the multiplier $1/(2\beta_+ + 1)$ for $2\beta_+ + 1 = 0$, which corresponds to $\lambda = \lambda_{**} = (5\sqrt{7} + 8)/(8 - \sqrt{7})$.

In the upper phase halfplane, i.e., for growing packets, the appearance of the phase curves is similar to that in Fig.2.

Below we describe the phase protraits of the second equation in (4.10) depending on the sign and magnitude of the coefficient $\beta_+(\lambda)$ and also in the limiting cases $\beta_+ = 0$, $\beta_+ = -1/2$.

1°. $\beta_+(\lambda) > 0$, $1 < \lambda < \lambda_*$. The behaviour of the phase trajectories in the region $a^\cdot < 0$ is similar to that described previously for rising packets. For the packet lifetime we have expression (5.3) with the change of variable $\beta_- \to \beta_+$.

2°. $\beta_+(\lambda) = 0$, $\lambda = \lambda_*$. The phase trajectories in this region are described by a one-parameter family of parabolas: $a^\cdot(a) = -(C_{21} - 2\alpha a)^{1/2}$, $C_{12} \geqslant 0$ is the parameter of the family.

The corresponding phase portrait is shown in Fig.3a. For the time of complete dissolution of a packet we obtain from (5.3) for $\beta_+ = 0$

$$T_* = \left(\frac{a_0}{2\alpha(\lambda_*)}\right)^{1/2} \int_0^1 \frac{d\tau}{(1-\tau)^{1/2}} \approx 1.58 \left(\frac{a_0}{g}\right)^{1/2}$$

3°. $0 > \beta_+(\lambda) > -1/2$, $\lambda_* < \lambda < \lambda_{**}$. The phase trajectories are described by the equality

$$a^\cdot(a) = -\left(C_{22} a^{-2\beta_+} - \frac{2\alpha}{2\beta_+ + 1} a\right)^{1/2}, \quad C_{22} \geqslant 0$$

(C_{22} is a parameter). Each curve of this family crosses the axis $a^\cdot = 0$ twice, and all the curves of the family pass through the origin.

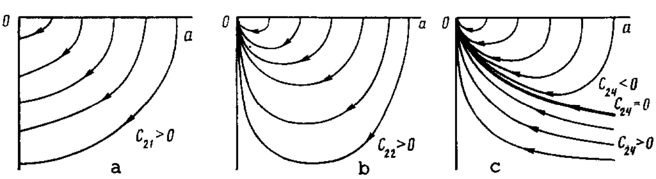

Fig.3

The location of the phase trajectories in the region $a^\cdot < 0$ is shown in Fig.3b. We see that in this case, contrary to the case $\beta_+ \geqslant 0$, the dissolution of the packet is non-monotonic. The velocity of the boundary $a^\cdot$ at first increases (remaining negative) until it reaches a certain maximum value. Then it starts decreasing, and at the moment when the packet vanishes the velocity of its boundary is $a^\cdot = 0$ (it was finite or infinite in the previous cases $\beta_+ = 0$ and $\beta_+ > 0$). For the dissolution time T_* we still have expression (5.3) with $\beta_+(\lambda)$ substituted for $\beta_-(\lambda)$.

4°. $\beta_+(\lambda) = -1/2$, $\lambda = \lambda_{**}$. In this limiting case, the family of phase curves has the form

$$a^\cdot(a) = -a^{1/2}(C_{23} - 2\alpha \ln a)^{1/2}$$

(C_{23} is a parameter of the family). The qualitative behaviour of the phase trajectories is identical to that in case 3°, with the sole difference that the parameter C_{23} takes all values in the interval $(-\infty, \infty)$ on the curves of the family. The packet lifetime is given by

$$T_*\left(\lambda = -\frac{1}{2}\right) = \left(\frac{a_0}{2\alpha}\right)^{1/2} \int_0^1 \frac{d\tau}{\tau^{1/2}(-\ln \tau)^{1/2}}$$

The improper integral in this expression is convergent.

5°. $\beta_+(\lambda) < -1/2, \lambda > \lambda_{**}$. The phase curves of the second equation (4.10) are described by the relationship

$$a^{\cdot}(a) = -\left(C_{24}a^{-2\beta_+} - \frac{2\alpha}{2\beta_+ + 1}a\right)^{1/2}$$

(C_{24} is a parameter). The curves with $C_{24} \geqslant 0$ cross the axis $a^{\cdot} = 0$ only at the origin. The curves of this family with $C_{24} < 0$ cross this axis twice, as in cases 3° and 4°. The curves of these two types are separated by the parabola

$$a^{\cdot 2}(a) = -\frac{2\alpha}{2\beta_+ + 1}a\ (C_{24} = 0)$$

The corresponding phase portrait is shown in Fig.3c. The velocity of the packet boundary at the moment of disappearance is zero, as in cases 3° and 4°. The dissolution of the packet is monotonic or non-monotonic, depending on whether the phase point $(a, a^{\cdot})$ is initially below (and on) the separating curve $C_{24} = 0$ or above the separating curve.

For the packet lifetime in this case we have

$$T_*(\lambda) = \left(-\frac{2\beta_+ + 1}{2\alpha(\lambda)}\right)^{1/2} \int_0^1 \frac{d\tau}{(\tau - \tau^{-2\beta_+(\lambda)})^{1/2}}$$

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